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Optimal Mortgage Refinancing: A Closed-Form Solution

We derive the first closed-form optimal refinancing rule: refinance when the current mortgage interest rate falls below the original rate by at least

$$\frac{1}{\psi}\left[\phi + W\left(-\exp\left(-\phi\right)\right)\right].$$

In this formula $W(.)$ is (the principal branch of) the Lambert W-function,

$$\psi = \frac{\sqrt{2\left(\rho+\lambda\right)}}{\sigma},$$
$$\phi = 1 + \psi\left(\rho+\lambda\right)\frac{\kappa/M}{\left(1-\tau\right)},$$

where ρ is the real discount rate, λ is the expected real rate of exogenous mortgage repayment, σ is the standard deviation of the mortgage rate, κ/M is the ratio of the tax-adjusted refinancing cost and the remaining mortgage value, and τ is the marginal tax rate. This expression is derived by solving a tractable class of refinancing problems. Our quantitative results closely match those reported by researchers using numerical methods.

JEL codes: G11, G21
Keywords: mortgage, refinance, option value.

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Households in the U.S. hold about $18 trillion in real estate assets.[1] Almost all home buyers obtain mortgages, and the total value of these mortgages is over $10 trillion, approaching the value of U.S. federal government debt held by the public. Decisions about mortgage refinancing are among the most important decisions that households make.[2]

Borrowers refinance mortgages to change the size of their mortgage and/or to take advantage of lower borrowing rates. Many authors have calculated the optimal refinancing differential when the household is not motivated by equity extraction considerations: Dunn and McConnell (1981a, 1981b), Dunn and Spatt (2005), Hendershott and van Order (1987), Chen and Ling (1989), Follain, Scott, and Yang (1992), Yang and Maris (1993), Stanton (1995), Longstaff (2005), Kalotay, Yang, and Fabozzi (2004, 2007, 2008), and Deng and Quigley (2006). At the optimal differential, the present value of the interest saved equals the sum of refinancing costs and the difference between an old "in-the-money" refinancing option that is given up and a new "out-of-the-money" refinancing option that is acquired.[3]

In the current paper, we derive a closed-form optimal refinancing rule. We begin our analysis by identifying an analytically tractable class of mortgage refinancing problems. We assume that the real mortgage interest rate and inflation follow Brownian motions, and the mortgage is structured so that its real value remains constant until an endogenous refinancing event or an exogenous Poisson repayment event. The Poisson parameter can be calibrated to capture the combined effects of moving events, principal repayment, and inflation-driven depreciation of the mortgage obligation.

The optimal refinancing solution depends on the discount factor, closing costs, mortgage size, the marginal tax rate, the standard deviation of the innovation in the mortgage interest rate, and the Poisson rate of exogenous real repayment. For calibrated choices of these parameters, the optimal refinancing differentials we derive range typically from 100 to 200 basis points. We compare our interest rate differentials with those computed numerically by Chen and Ling (1989), who do not make many of our simplifying assumptions. We find that the two approaches generate recommendations that differ by fewer than 10 basis points.

We provide two analytic solutions: a closed-form exact solution, which appears in the abstract, and a closed-form second-order approximation, which we refer to as the square-root rule. The closed-form exact solution makes use of Lambert's W-function (a little known but easily computable function that has only been actively studied in

1. Flow of Funds Accounts of the United States, Board of Governors of the Federal Reserve System, March, 2012.

2. Dickinson and Heuson (1994) and Kau and Keenan (1995) provide extensive surveys of the refinancing literature. See Campbell (2006) for a broader discussion of the importance of studying mortgage decisions by households.

3. Many papers have documented behavior by mortgage holders that appears to be at variance with predictions of the optimal refinancing model, including: Green and Shoven, (1986), Schwartz and Torous (1989, 1992, 1993), Giliberto and Thibodeau, (1989), Richard and Roll, (1989), Archer and Ling (1993), Stanton (1995), Archer, Ling, and McGill (1996), Hakim (1997), Green and LaCour-Little (1999), LaCour-Little (1999), Bennett, Peach, and Peristiani (2000, 2001), Hurst (1999), Downing, Stanton, and Wallace (2001), Hurst and Stafford (2004), Deng and Quigley (2006), and Chang and Yavas (2006).

the past 20 years). By contrast, our square-root rule can be implemented with any hand-held calculator. The square-root rule lies within 10–30 basis points of the exact solution for suitably calibrated choices of the parameters.

As described above, other authors have already *numerically* solved mortgage refinancing problems. Our contribution is to derive a *closed-form* solution.

Our closed-form solution has the disadvantage that we need to make several simplifying assumptions that reduce the realism of the problem. Most importantly, our model studies optimal refinancing for risk-neutral agents. Our model also assumes that the real mortgage interest rate is a random walk and that the expected rate of reduction in the real value of mortgage obligations is constant. On the other hand, our closed-form solution has several advantages. Our solution is transparent, tractable, and verifiable—it is not a numerical black box. It reveals the parametric and functional properties of an optimal refinancing policy. It can be integrated into analytic models. Finally, it can be used for pedagogy. In summary, our model makes several simplifying assumptions to obtain a payoff in tractability and transparency.

We also document a large gap between optimal refinancing and the advice given by almost all leading mortgage advisors. Advisors do not acknowledge or discuss the (option) value of waiting for interest rates to fall, and instead discuss a "break-even" present value rule: refinance if the present value of the interest savings is greater than or equal to the refinancing cost. Using our analytic framework we characterize the welfare losses of following such suboptimal break-even rules.

The paper has the following organization. Section 1 describes and solves a mortgage refinancing problem. Section 2 analyzes our refinancing result quantitatively and compares our results with the quantitative findings of other researchers. Section 3 compares our results with those of Chen and Ling (1989). Section 4 documents the advice of financial planners, and derives the welfare loss from following the present value rule. Section 5 applies our refinancing result to the aftermath of the current crisis by using our model to estimate the amount of interest payments that could be saved if people were able to refinance—over \$30 billion in real terms over the remainder of the mortgage. Section 6 concludes.

1. THE MODEL

In this section, we present a tractable continuous-time model of mortgage refinancing. The first subsection introduces the assumptions and notation. The next subsection summarizes the argument of the proof and reports the key results.

1.1 Notation and Key Assumptions

The real mortgage interest rate and the inflation rate. We assume that the real mortgage interest rate, r, and inflation rate, π, jointly follow Brownian motion. Formally,

$$dr = \sigma_r dz_r, \tag{1}$$

$$d\pi = \sigma_\pi dz_\pi, \tag{2}$$

where dz represents Brownian increments, and $\mathrm{cov}(dr, d\pi) = \sigma_{r\pi} dt$. Hence, the nominal mortgage interest rate, $i = r + \pi$ follows a continuous-time random walk.

The random walk approximation is problematic. Chen and Ling (1989), Follain, Scott, and Yang (1992), and Yang and Maris (1993) assume that the nominal mortgage interest rate follows a random walk. However, other authors assume that interest rates are mean reverting: for instance, Stanton (1995) and Downing, Stanton, and Wallace (2005). We adopt the random walk assumption because it allows us to considerably simplify the analysis.[4] In addition, Li, Pearson, and Poteshman (2004) argue that nominal interest rates are well approximated by random walks and that estimates showing mean reversion are biased because of failure to condition on observed minimums and maximums. Nevertheless, it is likely that there is some mean reversion in interest rates.[5]

In our model, the interest rate on a mortgage is fixed at the time the mortgage is issued. Our analysis focuses on the gap between the current nominal mortgage interest rate, $i = r + \pi$, and the "initial mortgage rate," $i_0 = r_0 + \pi_0$, which is the nominal mortgage interest rate at the time the mortgage was issued. Let x represent the difference between the current nominal mortgage interest rate and the initial mortgage rate: $x \equiv i - i_0$. This implies that

$$dx = \sqrt{\sigma_r^2 + \sigma_\pi^2 + 2\sigma_{r\pi}} dz \tag{3}$$

$$= \sigma dz, \tag{4}$$

where $\sigma \equiv \sqrt{\sigma_r^2 + \sigma_\pi^2 + 2\sigma_{r\pi}}$.

The mortgage contract. In a standard fixed-rate mortgage contract, nominal mortgage payments—the sum of interest payments and principal repayments—are constant over time. The real value of a mortgage obligation will decline for three different reasons: this contracted nominal principal repayment; a relocation, death, or other event that leads to repayment of the entire principal; and inflationary depreciation of the real value of the mortgage.

4. Another approach would be to make assumptions about the short-term riskless rate and derive the process for the mortgage rate from them. This approach has the attractive feature of allowing us to differentiate between the mortgage rate and the bank's cost of funds. Under the strong assumption that the real mortgage rate were equal to the sum of this real short-term rate and a constant wedge (which would ensure positive profits for the bank), the results of the analysis below would go through. Alternative assumptions that did not yield a random walk for the nominal mortgage interest rates would likely make a closed-form solution infeasible.

5. The random walk assumption is less tenable during times when interest rates are near the bounds of their historic ranges, as is presently true in the case of the U.S. However, current Japanese mortgage rates are over 2 percentage points lower than U.S. rates, suggesting that it is in principle possible for U.S. rates to fall from current levels. More generally, in recent years global long-term interest rates have seemed less responsive to variations in short-term policy interest rates, which suggests that mean-reverting behavior in policy rates may not translate into mean-reverting behavior in long-term rates.

To make the problem more analytically tractable, we counterfactually assume that the fixed-rate mortgage payments are structured so that the real value of the mortgage, M, remains constant until an exogenous and discrete mortgage repayment event, whose arrival rate we calibrate to capture all three reasons for the decline in the real value of the mortgage. This approach allows us to eliminate the real value of the mortgage as a state variable. Excluding these discrete repayment events, the continuous *flow* of real mortgage repayment is given by

$$\text{real flow of mortgage payments} = (r_0 + \pi_0 - \pi)M \tag{5}$$

$$= (i_0 - \pi)M. \tag{6}$$

We assume repayment events follow a Poisson arrival process, with hazard rate λ. In our calibration section, we show how to choose a value of λ that simultaneously captures all three channels of repayment: relocation, nominal principal repayment, and inflation. Hence, λ should be thought of as the *expected* exogenous rate of decline in the real value of the mortgage.[6] We note that this calibration will not capture the effects of large, discontinuous movements in the inflation process. To the extent that inflation is better characterized by the summation of a Brownian motion and a jump process, jumps in the inflation rate will lead to a discontinuous change in the real value of the mortgage not captured by λ. This is a limitation of the analysis required to obtain a closed-form solution.

Refinancing. The mortgage holder can refinance his or her mortgage at real (tax-adjusted) cost $\kappa(M)$. These costs include points and any other explicit or implicit transactions costs (e.g., lawyers fees, mortgage insurance, personal time). We define $\kappa(M)$ to represent the net present value (NPV) of these costs, netting out all allowable tax deductions generated by *future* deductions of amortized refinancing points. For a consumer who itemizes (and takes account of all allowable deductions), the formula for $\kappa(M)$ is provided in Appendix A.[7]

Our analysis translates costs and benefits into units of "discounted dollars of interest payments." Since $\kappa(M)$ represents the tax-adjusted NPV of closing costs, $\kappa(M)$ needs to be adjusted so that the model recognizes that one unit of κ is economically equal to $1/(1-\tau)$ dollars of current (fully and immediately tax deductible) interest payments, where τ is the marginal tax rate of the household. Hence, we multiply $\kappa(M)$ by $1/(1-\tau)$ and work with the normalized refinancing cost

$$C(M) = \frac{\kappa(M)}{1-\tau}.$$

6. The constancy of the hazard rate is needed to obtain the closed-form solution. In the calibration section, we discuss how violations of this assumption—for example, when few years are remaining in the mortgage—may lead our closed-form solution to differ from the optimal solution.

7. A borrower who itemizes is allowed to make the following deduction. If N is the term of the mortgage, then the borrower can deduct $\frac{1}{N}$ of the points paid for N years. If the mortgage is refinanced or otherwise prepaid, the borrower may deduct the remainder of the points at that time. Appendix A derives a formula for $\kappa(M)$.

If a consumer does not itemize, set $\tau = 0$ for both the calculation of $\kappa(M)$ and the calculation of $C(M)$.

Optimization problem. Mortgage holders pick the refinancing policy that minimizes the expected NPV of their real interest payments, applying a fixed discount rate, ρ. We assume that mortgage holders are risk neutral.

Summing up these considerations, the consumer *minimizes* the expected value of her real mortgage payments. Let value function $V(r_0, r, \pi_0, \pi, M)$ represent the expected value of her real mortgage payments. More formally, the instantaneous Bellman equation for this problem is given by

$$\begin{aligned}\rho V &= (r_0 + \pi_0 - \pi)M + \lambda M - \lambda V + \frac{E\,[dV]}{dt} \\ &= (r_0 + \pi_0 - \pi + \lambda)M - \lambda V + \frac{\sigma_r^2}{2}\frac{\partial^2 V}{\partial r^2} + \frac{\sigma_\pi^2}{2}\frac{\partial^2 V}{\partial \pi^2} + \sigma_{r\pi}\frac{\partial^2 V}{\partial r \partial \pi}.\end{aligned}$$

This Bellman equation can be derived with a standard application of stochastic calculus and Ito's Lemma. First-order partial derivatives do not appear in this expression, since r and π have no drift.[8]

At an endogenous refinancing event, the mortgage holder exchanges $V(r_0, r, \pi_0, \pi, M)$ for $V(r, r, \pi, \pi, M) + C(M)$. Hence, at an optimal refinancing event value matching will imply that

$$V(r_0, r, \pi_0, \pi, M) = V(r, r, \pi, \pi, M) + C(M).$$

Given our assumptions, an optimizing mortgage holder picks a refinancing rule that minimizes the discounted value of her mortgage payments. In other words, she picks a refinancing rule that minimizes V.

We next show that the second-order partial differential equation that characterizes V can be simplified.

1.2 Our Main Result

Since M is a constant, we can partial M out of the problem. This leaves four state variables: r_0, r, π_0, π.

The first step in the proof decomposes the value function $V(r_0, r, \pi_0, \pi)$. We define Z to be the discounted value of expected future payments conditional on the restriction that refinancing is disallowed. The Bellman equation for Z is given

8. The borrower's cost minimization problem can be directly written as choosing an optimal interest rate differential $x^* = r^* + \pi^* - (r_0 + \pi_0)$ to minimize the expected sum of the opportunity cost of not refinancing every instant until the interest rate differential $x = x^*$ and the refinancing cost $C(M)$. Let $T(x^*)$ denote the first time x hits x^*, and $\{oc(x_t) = -x_t M$ if $x_t \leq 0$, and 0 otherwise$\}$ the opportunity cost. Recalling that the effective discount rate is $\rho + \lambda$, and noting that the opportunity cost is incurred at all times until x^* is hit, but the refinancing cost is not incurred until x^* is hit, the cost minimization problem is: $\min_{x^*} E[\int_0^{T(x^*)} e^{-(\rho+\lambda)t} oc(x_t)dt + e^{-(\rho+\lambda)T(x^*)}C(M)]$.

by

$$\rho Z(r_0, r, \pi_0, \pi) = (r_0 + \pi_0 - \pi + \lambda)M - \lambda Z(r_0, r, \pi_0, \pi) + \frac{E\,[dZ]}{dt}.$$

It can be confirmed that the solution for Z is

$$Z(r_0, r, \pi_0, \pi) = \frac{(r_0 + \pi_0 - \pi + \lambda)\,M}{\rho + \lambda}. \tag{7}$$

It follows that Z can be reduced to a function of the state variable $-x + r$, which is equal to $r_0 + \pi_0 - \pi$.

We decompose V, by defining R as

$$R(r_0, r, \pi_0, \pi) \equiv Z(r_0, r, \pi_0, \pi) - V(r_0, r, \pi_0, \pi). \tag{8}$$

The function R represents the option value of being able to refinance. R can be expressed as a function of one state variable:

$$\begin{aligned} x &= i - i_0 \\ &= r + \pi - r_0 - \pi_0. \end{aligned}$$

This one-variable simplification can be derived with the following "replication" lemma.

LEMMA 1. *Replication*

$$R(r_0, r, \pi_0, \pi) = R(r_0 + \Delta, r + \Delta, \pi_0, \pi) \tag{9}$$

$$= R(r_0, r, \pi_0 + \Delta, \pi + \Delta) \tag{10}$$

$$= R(r_0, r + \Delta, \pi_0 + \Delta, \pi). \tag{11}$$

PROOF. Consider an agent in state $(r_0 + \Delta, r + \Delta, \pi_0, \pi)$. Let this agent replicate the refinancing strategy of an agent in state (r_0, r, π_0, π). In other words, refinance after every sequence of *innovations* in the Ito processes that would make the agent who started at (r_0, r, π_0, π) refinance. So the agent in state $(r_0 + \Delta, r + \Delta, \pi_0, \pi)$ will generate refinancing choices valued at $V(r_0, r, \pi_0, \pi) + \Delta M/(\rho + \lambda)$ (where $\Delta M/(\rho + \lambda)$ is the increase in the real value of mortgage payments, discounted at the subjective rate of time discount ρ augmented by the exogenous probability of mortgage repayment λ). Hence,

$$V(r_0 + \Delta, r + \Delta, \pi_0, \pi) \leq V(r_0, r, \pi_0, \pi) + \frac{\Delta M}{\rho + \lambda}.$$

Likewise, we have

$$V(r_0, r, \pi_0, \pi) \leq V(r_0 + \Delta, r + \Delta, \pi_0, \pi) - \frac{\Delta M}{\rho + \lambda}.$$

Combining these two inequalities, and substituting equations (7) and (8), yields equation (9). We now repeat this type of argument for other cases. By replication,

$$V(r_0, r, \pi_0 + \Delta, \pi + \Delta) \leq V(r_0, r, \pi_0, \pi)$$
$$V(r_0, r, \pi_0, \pi) \leq V(r_0, r, \pi_0 + \Delta, \pi + \Delta).$$

Combining these two inequalities, we have equation (10). By replication,

$$V(r_0, r + \Delta, \pi_0 + \Delta, \pi) \leq V(r_0, r, \pi_0, \pi) + \frac{\Delta M}{\rho + \lambda}$$
$$V(r_0, r, \pi_0, \pi) \leq V(r_0, r + \Delta, \pi_0 + \Delta, \pi) - \frac{\Delta M}{\rho + \lambda}.$$

Before refinancing, the perturbed agent pays ΔM more (the inflation rate at which the perturbed agent borrowed is $\pi_0 + \Delta$ rather than π_0). After refinancing, the perturbed agent pays ΔM more (the real interest rate at which the perturbed agent refinances is $r + \Delta$ rather than r). Combining the two inequalities, we have equation (11). □

The lemma implies that these equalities hold everywhere in the state space:

$$\frac{\partial R}{\partial r} = \frac{\partial R}{\partial \pi} = -\frac{\partial R}{\partial r_0} = -\frac{\partial R}{\partial \pi_0}.$$

This in turn implies that $R(r_0, r, \pi_0, \pi)$ can be rewritten as $R(x)$.

We will show that the solution of R can be expressed as a second-order ordinary differential equation with three unknowns: two constants in the differential equation and one free boundary. To solve for these three unknowns we need three boundary conditions. We will exploit a value-matching constraint that links R the instant before refinancing at $x = x^*$ and the instant after refinancing (when $x = 0$):

$$R(x^*) = R(0) - C(M) - \frac{x^* M}{\rho + \lambda}.$$

We will also exploit smooth pasting at the refinancing boundary

$$R'(x^*) = -\frac{M}{\rho + \lambda}.$$

Finally, $\lim_{x \to \infty} R(x) = 0$, since the option value of refinancing vanishes as the interest differential gets arbitrarily large. See Lemma 2 in Appendix B for a derivation of the first two boundary conditions.

The following theorem characterizes the optimal threshold, x^*, and the value functions. The threshold rule is expressed in x, the difference between the current nominal interest rate, i, and the nominal interest rate of the mortgage, i_0.

THEOREM 1. *Refinance when*

$$i - i_0 \leq x^* \equiv \frac{-1}{\psi}[\phi + W(-\exp(-\phi))], \tag{12}$$

where $W(.)$ is the principal branch of the Lambert W-function,

$$\psi = \frac{\sqrt{2(\rho+\lambda)}}{\sigma},$$

$$\phi = 1 + \psi(\rho+\lambda)\frac{\kappa/M}{(1-\tau)}.$$

When $x > x^*$ the value function is

$$V(r_0, r, \pi_0, \pi) = -Ke^{-\psi x} + \frac{(i_0 - \pi + \lambda)M}{\rho+\lambda}, \tag{13}$$

where K[9] *is given by*

$$K = \frac{Me^{\psi x^*}}{\psi(\rho+\lambda)}. \tag{14}$$

The option value of being able to refinance is $Ke^{-\psi x}$ when $x > x^*$.

PROOF. We can express V as

$$V(x, r) = \frac{(-x + r + \lambda)M}{\rho+\lambda} - R(x).$$

Using Ito's Lemma, derive a continuous-time Bellman equation for V

$$\rho V = (-x + r)M + \frac{\sigma^2}{2} \cdot \frac{\partial^2 V}{\partial x^2} + \lambda(M - V). \tag{15}$$

Substituting for V yields

$$(\rho+\lambda)\left(\frac{(-x + r + \lambda)M}{\rho+\lambda} - R\right) = (-x + r + \lambda)M - \frac{\sigma^2}{2}R''.$$

This simplifies to

$$(\rho+\lambda)R = \frac{\sigma^2}{2}R''. \tag{16}$$

9. K has an equivalent solution, $K = -(e^{-\psi x^*} - 1)^{-1}(\frac{x^* M}{\rho+\lambda} + C(M))$.

The original value function V has been eliminated from the analysis, as has the variable r. The option value $R(x)$ has a solution of the form $R(x) = Ke^{-\psi x}$, with exponent

$$\psi = \frac{\sqrt{2(\rho + \lambda)}}{\sigma}.$$

We pick $\psi > 0$ to satisfy the limiting boundary condition ($\lim_{x^* \to \infty} R(x^*) = 0$). The remaining two parameters, K and x^*, solve the system of equations derived from the value-matching and smooth-pasting conditions

$$Ke^{-\psi x^*} = K - C(M) - \frac{x^* M}{\rho + \lambda} \tag{17}$$

$$-\psi K e^{-\psi x^*} = -\frac{M}{\rho + \lambda}. \tag{18}$$

We use the smooth-pasting condition to solve for K and substitute it back into the value-matching condition. Hence,

$$K = \frac{1}{\psi} \frac{M e^{\psi x^*}}{\rho + \lambda}, \tag{19}$$

yielding

$$\frac{1}{\psi} \frac{M}{\rho + \lambda} = \frac{1}{\psi} e^{\psi x^*} \frac{M}{\rho + \lambda} - C(M) - \frac{x^* M}{\rho + \lambda}. \tag{20}$$

Multiplying through by the inverse of the left-hand side yields

$$e^{\psi x^*} - \psi x^* = 1 + \frac{C(M)}{M} \psi(\rho + \lambda). \tag{21}$$

Set $k = 1 + C(M)\psi(\rho + \lambda)/M$ in Lemma 3 (Appendix B) to yield the closed-form expression for x^* in the statement of the theorem. □

The principal branch of the Lambert W-function, which appears in the solution, is the inverse function of $f(x) = xe^x$ for $x \geq -1$. Hence, $z = W(z)e^{W(z)}$. Although its origins can be traced to Johann Lambert and Leonhard Euler in the eighteenth century, the function has only been extensively studied in the past 20 years. It has since been shown to be useful in solving a wide variety of problems in applied mathematics, and is built into common software, including Maple, Mathematica, and Matlab. For more information on the function and its uses, see Corless et al. (1996) and Hayes (2005).

We also study an additional threshold value at which the reduction in the present value (PV) of future interest payments (assuming no more refinancing) is exactly offset by the cost of refinancing, $C(M)$. We refer to this as the PV break-even threshold.[10]

DEFINITION 1. *The PV break-even threshold, x^{PV}, is defined as*

$$-\frac{x^{PV}M}{\rho+\lambda} = C(M). \tag{22}$$

Intuitively, the PV break-even threshold is the point at which the expected interest payments saved from an immediate and final refinancing, $-xM/(\rho+\lambda)$, exactly offset the tax-adjusted cost of refinancing, $C(M)$.

1.3 Second-Order Expansion

Our closed-form (exact) solution for the optimal refinancing differential requires calls to the Lambert W-function. We also provide an alternative approximation that does not involve the Lambert W-function.

The proof of the main theorem derives an implicit solution for x^* (equation (21)), which can be written as

$$f(x^*) = e^{\psi x^*} - \psi x^* - 1 - \psi(\rho+\lambda)\frac{C(M)}{M} = 0.$$

A second-order Taylor series approximation to $f(x^*)$ at $x^* = 0$ is given by

$$\begin{aligned} f(x^*) &\approx f(0) + f'(0)x^* + \frac{1}{2}f''(0)x^{*2} \\ &= -\psi(\rho+\lambda)\frac{C(M)}{M} + 0\cdot x^* + \frac{1}{2}\psi^2 x^{*2}. \end{aligned}$$

Setting this to zero and solving for x^* (picking the negative root) yields an approximation that we refer to as the *square-root rule,*

$$x^* \approx -\sqrt{\frac{\sigma\kappa}{M(1-\tau)}\sqrt{2(\rho+\lambda)}}.$$

We evaluate the practical accuracy of this approximation in the calibration section.[11] We also evaluate a third-order approximation, which is given by an implicit cubic equation.

10. Follain and Tzang (1988) also derive this differential. They note that since it ignores the option to refinance, this differential represents a lower bound to the refinancing decision; they also note that calculating the option value is complicated.

11. Not all of the limit properties of the second-order approximation match those of the exact solution. In particular, as the standard deviation of the mortgage rate, σ, goes to zero, the second-order approximation also goes to zero, while the exact solution goes to the NPV threshold. Because of this, at low values of σ, the NPV threshold is a better approximation to the optimal threshold than is the second-order approximation. Since the NPV threshold is also easily calculable, a better refinancing rule than simply using the second-order approximation alone is to refinance when $x < \min\{-\sqrt{\sigma\frac{C(M)}{M}\sqrt{2(\rho+\lambda)}}, -(\rho+\lambda)\frac{C(M)}{M}\}$.

2. CALIBRATION

We begin by illustrating the model's predictions for the optimal threshold value x^*. We numerically solve equation (12)—the exact solution of the optimal refinancing problem—with typical values of parameters ρ, τ, $\kappa(M)$, σ, and λ. Analytic derivatives of the threshold with respect to these parameters are provided in Appendix C. We also provide a web calculator that readers can use to evaluate any calibration of interest.[12]

For our first illustrative analysis, we choose a 5% real discount rate, $\rho = 0.05$. We assume a 28% marginal tax rate, $\tau = 0.28$.[13] We assume transactions costs of 1 point and \$2,000; $\kappa(M)$ is given by the formula in Appendix A (e.g., $\kappa(M) = 0.01M + 2000$ if $\tau = 0$). The fixed cost (\$2000) reflects a range of fees including inspection costs, title insurance, lawyers fees, filing charges, and nonpecuniary costs like time.[14] Using historical data, we estimate that the annualized standard deviation of the mortgage interest rate is $\sigma = 0.0109$.[15]

Finally, we need to calibrate λ, the expected real repayment rate of the mortgage. We need to calculate the value of λ that corresponds to a *realistic* mortgage contract—one in which there are three forms of repayment: first, a probability of exogenous repayment (due, for example, to a relocation); second, principal payments that reduce the real value of the mortgage; third, inflation that reduces the real value of the mortgage. Formally, consider a household with a mortgage with a contemporaneous real (annual) mortgage payment of p, remaining principal M, an original nominal interest rate of i_0, and a μ hazard of relocation (implying that $1/\mu$ is the expected time until the next move). We will consider an environment with current inflation π. For this mortgage, the expected (flow) value of the exogenous decline in the real mortgage obligation is

$$\mu M + (p - i_0 M) + \pi M.$$

The term in parentheses corresponds to contracted principal repayment.[16] The last term represents inflation eroding the real value of the mortgage. Using this formula,

12. http://www.nber.org/mortgage-refinance-calculator.

13. In the 2010 tax code, the 28% marginal tax rate applies to joint filings for households with joint income between \$137,300 and \$209,250, and to filings for single households with income between \$82,400 and \$171,850.

14. See Federal Reserve Board and Office of Thrift Supervision (1996), Caplin, Freeman, and Tracy (1997), Danforth (1999), LaCour-Little (2000), and Chang and Yavas (2006) for data on transactions costs.

15. The standard deviation for monthly differences of the Freddie Mac 30-year mortgage rate from April 1971 to February 2004 is 0.00315, implying an annualized standard deviation of $\sigma = \sqrt{12} \times 0.00315 = 0.0109$. By comparison, taking annual differences yields an average standard deviation of $\sigma = 0.0144$. These results are consistent with our decision to model interest rate innovations as i.i.d.

16. Note that in the case of an interest-only mortgage, in which there are no principal repayments, this term will drop out, but the rest of our analysis will carry through.

TABLE 1

REFINANCING DIFFERENTIALS IN BASIS POINTS BY SOLUTION METHOD

Mortgage	Exact optimum	Second order	Third order	PV rule
\$1,000,000	107	97	109	27
\$500,000	118	106	121	33
\$250,000	139	123	145	44
\$100,000	193	163	211	76

we can calibrate the value of λ,

$$\begin{aligned}\lambda &= \mu + \left(\frac{p}{M} - i_0\right) + \pi \\ &= \mu + \left(\frac{p_{\text{nominal}}}{M_{\text{nominal}}} - i_0\right) + \pi.\end{aligned}$$

In practice, it will be easier for many households to use the latter "nominal" version of the formula since households know p_{nominal} and M_{nominal}, the nominal analogs of p and M.[17]

We can also solve for the key terms in the equations above using formulae for a standard fixed rate mortgage. In this case, the calibration for λ is

$$\lambda = \mu + \frac{i_0}{\exp[i_0\Gamma] - 1} + \pi,$$

where Γ is the remaining life (in years) of the mortgage. See Appendix D for this derivation.

Assume that the household has a 10% chance of moving per year, so $\mu = 10\%$,[18] and the expected duration of staying in the house is 10 years. Assume that $i_0 = 0.06$, $\pi = 0.03$, and $\Gamma = 25$ years, then $\lambda = 0.147$.

Table 1 reports the optimal refinancing differentials calculated with our model for the calibration summarized above. We report the exact optimal rule, the second- and third-order approximations to the optimal rule, and the (suboptimal) present value rule. We calculate the refinancing differentials for mortgage sizes (M) of \$1,000,000, \$500,000, \$250,000, and \$100,000.

The optimal refinancing threshold increases as mortgage size decreases, since interest savings from refinancing scale proportionately with mortgage size but part of the refinancing cost is fixed (\$2,000). The second-order approximation deviates by 10–30 basis points from the exact optimum. The third-order approximation deviates

17. This calibration is only an approximation, since the calibration formula will change over time (whereas λ is constant in the model from Section 1).

18. Hayre, Chaudhary, and Young (2000) estimate that 5–7% of single-family homes turnover per year.

TABLE 2

Optimal Refinancing Differentials in Basis Points by Marginal Tax Rate τ

	τ						
Mortgage	0%	10%	15%	25%	28%	33%	35%
$1,000,000	99	101	103	106	107	109	110
$500,000	108	111	113	117	118	121	122
$250,000	124	129	131	137	139	143	145
$100,000	166	174	178	189	193	199	202

TABLE 3

Optimal Refinancing Differentials in Basis Points by Expected Real Rate of Repayment λ

	λ		
Mortgage	0.114	0.147	0.247
$1,000,000	101	107	122
$500,000	112	118	136
$250,000	131	139	161
$100,000	180	193	227

by only 2–18 basis points from the exact optimum. The NPV rule, by contrast, deviates by 80–117 basis points from the exact optimum.

Table 2 presents results for the six different marginal tax rates that were in effect under the tax code in 2006.

The optimal differentials rise as the marginal tax rate rises, since interest payments are tax deductible but refinancing costs are not.

Table 3 reports the consequences of varying λ, the expected real rate of repayment, due to differences in the expected time of moving.[19] We consider cases in which the expected time to the next move is 5 years ($\mu = 0.20$), 10 years ($\mu = 0.10$), and 15 years ($\mu = 0.0\overline{6}\overline{6}$), corresponding to values for λ of 0.247, 0.147, and 0.114, respectively.

As expected, a higher hazard rate of prepayment raises the optimal interest rate differential, since the effective amount of time over which the lower interest savings will be realized is smaller.

Another factor that can have a large effect on λ is the number of years remaining on the mortgage loan, Γ. When Γ becomes small, the calibration for λ simplifies to $\lambda \approx \mu + \frac{1}{\Gamma} + \pi$. Keeping the expected time of relocation at 10 years and the average inflation rate at 3%, the value of Γ that corresponds to the case of $\lambda = 0.247$ presented above is about 8.5 years. Thus, as one would expect, the optimal refinancing

19. Unless otherwise specified, we now return to our earlier assumption of a marginal tax rate of 28%.

TABLE 4

Optimal Refinancing Differentials in Basis Points by Fee Size

	$F + fM$		
Mortgage	$2000 + 0.01$M$	$1000	$1000, PV
$1,000,000	107	32	3
$500,000	118	45	5
$250,000	139	66	11
$100,000	193	108	27

differential increases with decreased remaining time on the mortgage because a lower mortgage rate is needed to cover the NPV of the refinancing cost. We should note that for mortgages of very short remaining duration, the assumption of a constant hazard of repayment will break down, since repayment will occur with certainty in a short time. In such cases, a more realistic model would likely imply a higher refinancing differential than ours, though the difference may not be large. For example, in the case where one year remains on the mortgage, our model implies that one should not refinance a $100,000 mortgage unless the interest rate has dropped by over 500 basis points.

Table 4 reports the optimal differential assuming a refinancing cost of only $1,000, which is of interest because of the wider availability of low-cost refinancings. For comparison, we also report the differentials predicted by the PV rule at a refinancing cost of $1,000.

Reducing the costs substantially reduces the optimal interest rate differentials. The differentials implied by the PV rule also decline.

3. COMPARISON WITH CHEN AND LING (1989)

We now compare the refinancing differentials implied by our model and those reported by Chen and Ling (1989). Chen and Ling calculate optimal differentials for a model in which the log one-period nominal interest rate follows a random walk, the time of exogenous prepayment (or the expected holding period) is known with certainty, and the real mortgage principal is allowed to decline over time because of inflation and continuous principal repayment. Chen and Ling assume that preferences are risk neutral. Chen and Ling use numerical methods to solve the resulting system of partial differential equations.

In contrast to their analysis, we make a simplifying assumption that allows us to obtain an *analytic* solution to a closely related mortgage refinancing problem.[20]

20. In one way, our paper adds greater realism as compared to previous work. We account for the differential tax treatment of mortgage interest payments and refinancing costs. Refinancing costs are not tax deductible (unlike the closing costs on an originating mortgage).

As explained above, we assume that the mortgage is structured so that its real value remains constant. This allows us to avoid tracking a changing value of time to maturity and a changing remaining mortgage balance. In contrast to our approach, Chen and Ling's model directly incoporates the effects of the finite life of the mortgage contract and principal repayment.

To bring our model in line with theirs, our parameter λ is calibrated to capture the joint effects of moving, principal repayments, and inflation. Hence, λ is set to capture the three ways that the expected real value of the mortgage declines over time.

To calibrate our model to match the setup in Chen and Ling, we set $\lambda = 0.173$ to account for (i) an 8-year expected holding period ($1/\mu = 8$, so $\mu = 0.125$), (ii) a long-run inflation forecast (in 1989) of 4% ($\pi = 0.04$), and (iii) a principal repayment rate of 0.8% at the beginning of a 30-year mortgage. We set the discount rate to be 4%, $\rho = 0.04$, matching Chen and Ling's assumption of an 8% *nominal* interest rate. Chen and Ling's random walk assumption for the log short-term interest rate allows us to compute the implied standard deviation for the 30-year mortgage rate (see Appendix E). We calculate an implied standard deviation for the innovations of the 30-year mortgage rate of $\sigma = 0.012$. Finally, to match the analysis of Chen and Ling we assume a zero marginal tax rate.[21]

Chen and Ling's baseline calculations exclude the possibility of subsequent refinancings. But their analysis enables us to compute the additional points that would be necessary to buy a new refinancing option when the original mortgage is refinanced. There are two such cases that are analyzed in Chen and Ling.

With a refinancing cost of 2 points (without a new option to refinance), 2.24 additional points are charged to purchase the right to refinance again, implying total points of 4.24 (see panel 1, column 3, in Table 1 of Chen and Ling 1989). For this case, Chen and Ling calculate an optimal refinancing differential of 228 basis points, while we calculate an optimal refinancing differential of 218 basis points, a difference of 10 basis points.

With a refinancing cost of 4 points (without a new option to refinance), 1.51 additional points will be charged to purchase the right to refinance again, implying total points of 5.51 points (see panel 1, column 3, in Table 1 of Chen and Ling 1989). For this case, Chen and Ling calculate an optimal refinancing differential of 256 basis points, while we calculate an optimal differential of 255 basis points, a difference of 1 basis point.[22]

21. We take results from the middle columns of Chen and Ling's (1989) Table 2. For consistency with our framework, we consider cases from Chen and Ling in which the interest rate process has no drift.

22. The second-order approximations yield refinancing differentials of 182 and 207, differing from Chen and Ling's values by 46 and 48 basis points. These results reflect the general deterioration of the approximation as refinancing costs become very large.

4. FINANCIAL ADVICE

Households considering refinancing use many different sources of advice, including mortgage brokers, financial planners, financial advice books, and websites. In this section, we describe the refinancing rules recommended by the top 25 leading books and websites. We find that none of the sources of financial advice in our sample provide a calculation of the optimal refinancing differential. Instead, the advisory services in our sample offer the break-even PV rule as the only theoretical benchmark. Most of the advice boils down to the following necessary condition for refinancing—only refinance if you can recoup the closing costs of refinancing in reduced interest payments.

First, we sampled books that were on top-10 sales lists at the Amazon and Barnes & Noble websites (see the online appendix for a detailed description of our sampling method and findings).[23] Of the 15 unique books in our sample, 13 provided a break-even calculation of some sort. Most of the 15 books also provided some rules of thumb (e.g., "wait for an interest differential of 200 basis points" or "only refinance if you can recoup the closing costs within 18 months").

For websites, we entered the words *mortgage refinancing advice* into Google and examined the top 12 sites that offered information on refinancing. Two of these sites suggest a fixed interest rate differential of 1.5% to 2% and recommend refinancing only if the borrower plans to stay in the house for at least 3–5 years. One of the sites provides a monthly savings calculator, while seven of the sites provide a refinancing calculator based on the PV break-even criterion. The remaining three sites did not provide a refinancing calculator but still recommend break-even calculations.

None of the 15 books and 10 websites in our sample discuss (or quantitatively analyze) the value of waiting due to the possibility that interest rates might continue to decline.

Although our sampling procedure above did not identify books or websites that mentioned option value considerations, that does not mean that such sites do not exist. Indeed, there is at least one website that does implement a numerical option value solution (http://www.kalotay.com; Kalotay, Yang, and Fabozzi 2004, 2007, 2008).

Finally, market data also show that many households *did* refinance too close to the PV break-even rule during the last 15 years; see, for example, Chang and Yavas (2009) and Agarwal, Driscoll, and Laibson (2004).

4.1 How Suboptimal Is the PV Rule?

To measure the suboptimality of the PV rule, we consider an agent that starts life with state variable $x = 0$ (a new mortgage). We calculate the expected cost of using an arbitrary refinancing differential, x^H, instead of using the *optimal* refinancing rule specified in Theorem 1.

23. http://www.nber.org/mortgage-refinance-calculator/appendix.py.

Proposition 1. *The expected discounted loss as a fraction of the mortgage size from using an arbitrary heuristic rule instead of using the optimal rule is given by*

$$\frac{Loss}{M} = \frac{\dfrac{C(M)}{M} + \dfrac{x^*}{\rho + \lambda}}{1 - e^{-\psi x^*}} - \frac{\dfrac{C(M)}{M} + \dfrac{x^H}{\rho + \lambda}}{1 - e^{-\psi x^H}} \tag{23}$$

$$= \frac{e^{\psi x^*}}{\psi(\rho + \lambda)} - \frac{\dfrac{C(M)}{M} + \dfrac{x^H}{\rho + \lambda}}{1 - e^{-\psi x^H}}, \tag{24}$$

where x^H is the heuristic threshold rule. This implies that the expected discounted loss as a fraction of the mortgage size from using the suboptimal NPV rule instead of using the optimal rule is given by

$$\frac{Loss}{M} = \frac{e^{\psi x^*}}{\psi(\rho + \lambda)}. \tag{25}$$

Proof. The loss is equal to the difference between the value function associated with the optimal rule and the value function associated with the alternative rule. The value function for the optimal rule is given in the statement of the main theorem. Since the interest payment term is the same for both the optimal and suboptimal rules, the difference in value functions will be equal to the difference in option value expressions. For both the suboptimal and approximate rules, the value-matching condition still applies, but with x^* replaced with the suboptimal differentials specified by the alternative rule, x^H.

Following the line of argument in the proof of our main theorem, the option value function, $R(x)$, has a solution of the form $R(x) = Ke^{-\psi x}$. The parameter K is derived from the value-matching condition,

$$Ke^{-\psi x^H} = K - C(M) - \frac{x^H M}{\rho + \lambda}, \tag{26}$$

implying

$$K = \frac{C(M) + \dfrac{x^H M}{\rho + \lambda}}{1 - e^{-\psi x^H}}. \tag{27}$$

So the difference in value functions is given by

$$\frac{Loss}{M} = \left[\frac{\dfrac{C(M)}{M} + \dfrac{x^*}{\rho + \lambda}}{1 - e^{-\psi x^*}} - \frac{\dfrac{C(M)}{M} + \dfrac{x^H}{\rho + \lambda}}{1 - e^{-\psi x^H}} \right] e^{-\psi x}$$

TABLE 5

EXPECTED LOSSES IN DISCOUNTED DOLLARS FROM USING THE PV AND SQUARE-ROOT RULES

Mortgage	Loss (PV rule)	Loss (square-root rule)
\$1,000,000	\$47,531	\$189
\$500,000	\$22,244	\$123
\$250,000	\$9,859	\$92
\$100,000	\$2,897	\$80

TABLE 6

EXPECTED LOSSES AS A PERCENT OF MORTGAGE FACE VALUE FROM USING THE PV AND SQUARE-ROOT RULES

Mortgage	Loss (PV rule)	Loss (square-root rule)
\$1,000,000	4.75%	0.02%
\$500,000	4.45%	0.02%
\$250,000	3.94%	0.04%
\$100,000	2.90%	0.08%

$$= \left[\frac{e^{\psi x^*}}{\psi(\rho+\lambda)} - \frac{\dfrac{C(M)}{M} + \dfrac{x^H}{\rho+\lambda}}{1 - e^{-\psi x^H}} \right] e^{-\psi x}.$$

Note that $x^H = x^{PV}$ implies that $\frac{C(M)}{M} + \frac{x^H}{\rho+\lambda} = 0$, and hence

$$\frac{Loss}{M} = \frac{\dfrac{C(M)}{M} + \dfrac{x^*}{\rho+\lambda}}{1 - e^{-\psi x^*}} e^{-\psi x} = \frac{e^{\psi(x^*-x)}}{\psi(\rho+\lambda)}.$$

Set $x = 0$, to reflect the perspective of an agent with a newly issued mortgage. □

Note that the loss from following the PV rule is equal to the option value of the ability to refinance, evaluated for a new mortgage. By ignoring the existence of the option value, the PV rule creates a loss equal in size to the option value.

Using the same calibration assumptions that were used in Section 2, Tables 5 and 6 report the economic losses of using the PV rule and the second-order rule instead of the exactly optimal rule.

4.2 Other Rules of Thumb

Some advisers also refer to a rule of thumb in which borrowers are encouraged to refinance when the interest rate has dropped by 200 basis points. We have also heard more recently of a revised 100 basis point rule of thumb. Both rules generally, though not always, imply refinancings at bigger differentials than those implied by the PV rule. However, our simulations show that the optimal refinancing differential can vary quite substantially by expected holding period and refinancing cost, among other parameters. Hence a "one-size-fits-all" rule will lead to substantial welfare losses.

5. IMPLICATIONS FOR THE AFTERMATH OF THE FINANCIAL CRISIS OF 2008

Over the past several years, researchers and commentators have argued that lax mortgage underwriting standards were a key cause of the financial crisis of 2008 (for an example of the former, see Keys et. al. 2010). In response to the macroeconomic consequences of the crisis, the Federal Reserve and other central banks around the world lowered short-term interest rates in order to increase aggregate demand by boosting interest-sensitive spending. Normally, this effect would work through the housing market in two ways: lower interest rates would make it easier for potential new homeowners to buy houses, and for existing homeowners to refinance to save on interest payments and consume some of the proceeds. However, in the aftermath of the crisis many lenders have tightened standards on both new mortgage origination and refinancings, and Fannie Mae and Freddie Mac have also increased their requirements for buying and securitizing mortgages.[24] While such tightenings in standards are in part a natural reaction to the laxity of standards prior to the crisis and likely help benefit financial stability, they also partly undercut the beneficial effects of lower interest rates by leaving many borrowers unable to obtain new mortgages or to refinance.

We can use our optimal refinancing model to obtain a rough estimate of the extent to which U.S. borrowers are currently paying higher interest rates than they otherwise would be due to such higher standards. Figure 1 shows the distribution of the dollar amount outstanding, by interest rate, on existing prime 30-year fixed-rate mortgages in the Loan Performance Service (LPS) data set.[25] For each interest rate category, we break up the distribution of interest rates for those borrowers with a loan-to-value (LTV) ratio of less than 80%, those with LTVs of between 80% and 100%, and those that are underwater (i.e., with LTVs of above 100%).

Assuming that the current mortgage interest rate for a 30-year fixed-rate loan is 4%, our model would suggest that mortgage holders with interest rates of 5.5% or above should optimally refinance their mortgages (since our optimal differentials are around 150 basis points). Of that group, the small number of borrowers with mortgage rates of above 6.5% would likely have seen a 150 point drop prior to the onset of the crisis and the tightening of mortgage standards, but chose not to refinance. Their

24. For evidence on the former, see the Federal Reserve Board's Senior Loan Officer Opinion Survey on Bank Lending Practices, which reported substantial numbers of banks tightening standards and terms on residential loans for many quarters after the crisis.

25. This data set, for the period 2011:Q1, contains information on 64% of all mortgages outstanding. See Agarwal, Chang, and Yavas (2012) for more information.

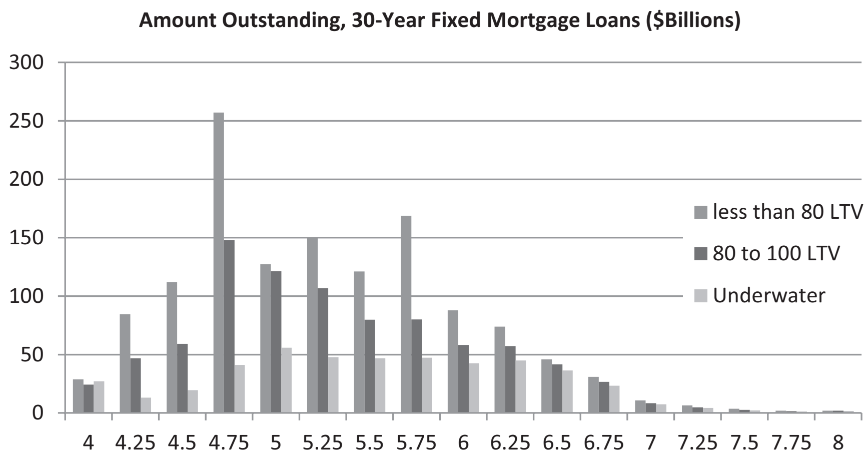

Fig. 1. Amount Outstanding (in Billions of Dollars) of 30-Year Fixed-Rate Prime Mortgage Loans, by Mortgage Interest Rate and Loan-to-Value (LTV) Ratio.
Note: Date: 2011:Q1.
Source: Loan Performance Service (LPS).

reasons for refinancing may thus lie outside our model, and therefore we focus our analysis on those with current mortgage rates of between 5.5% and 6.5%.[26] We also limit our analysis to those borrowers with LTVs of between 80% and 100%. Those borrowers with LTVs of less than 80% are likely to be able to refinance even given the new tightness in standards but are choosing not to do so for reasons not in our model. Underwater borrowers are likely unable to refinance given the new tightness in standards, but it is more difficult to argue that such borrowers should be allowed to do so than for those borrowers with lower LTVs, since such borrowers will likely still be more likely to default.

Borrowers with LTVs of between 80% and 100%—about $317 billion in loans on 2.5 million mortgages—are likely not able to refinance, even if they wanted to do so, due to the tightening in credit standards. The single year saving is about $6 billion if such borrowers refinanced to 4%.[27] Discounting at the rate $\rho + \lambda = 0.197$ to calculate the expected real value of savings over the life of the mortgage yields about $31 billion in savings.[28]

26. The industry has identified groups of borrowers who do not appear to refinance regardless of the drop in mortgage interest rates, and termed them "woodheads."

27. The savings rise to $7.5 billion if we include those with mortgage rates of more than 6.5%, to $16.8 billion if we also include those borrowers with LTVs of less than 80%, and $21.1 billion if we include those with underwater mortgages.

28. Or about $100 billion if all borrowers with interest rates of over 5.5%, regardless of LTV, are included.

6. CONCLUSION

Optimal mortgage refinancing rules have been calculated previously by numerically solving a system of partial differential equations. This paper derives the first *closed-form* solution to a mortgage refinancing problem.

Our closed-form solution has the disadvantage that we need to make several simplifying assumptions—most importantly, risk neutrality, a real mortgage interest rate that follows a random walk, and a constant expected rate of decline in the real value of mortgage obligations. On the other hand, closed-form solutions are transparent, tractable, and easily verified—they are not a numerical black box. The functional contribution of each parameter to the solution is immediately apparent. One can easily make comparative statics calculations and can compute other important magnitudes—for example, welfare calculations—without resorting to computational methods. Closed-form solutions are useful for pedagogy and as building blocks for more complex models. Analytic approximations are frequently used in the natural sciences and engineering to improve our understanding of "exact" numerical models.[29]

Our derived refinancing rule requires specification of the discount rate, the expected real rate of exogenous mortgage repayment (including the effects of moving, principal repayment, and inflation), the annual standard deviation of the mortgage rate, the ratio of the tax-adjusted refinancing cost and the remaining value of the mortgage, and the marginal tax rate. All of these variables are easy to calibrate, including the rate of exogenous mortgage prepayment, which can be obtained from the annual probability of relocating, the ratio of total mortgage payments to the remaining value of the mortgage, the number of years remaining in the mortgage, the initial nominal mortgage interest rate, and the current inflation rate.

We analyze both the exact solution of our mortgage refinancing problem and a useful approximation to that solution. We show that a second-order Taylor expansion yields a square-root rule for optimal refinancing.

We show that many leading sources of financial advice do not discuss (formally or informally) option value considerations. Advisory services typically discuss the PV rule: refinance only if the PV of the interest saved is at least as great as the direct cost of refinancing. Compared to the optimal refinancing rule, the PV rule generates expected discounted losses of over \$85,000 on a \$500,000 mortgage.

Finally, we use our refinancing rule to try to estimate the size of interest payments saved for current (as of 2011:Q1) borrowers who in principle should refinance, but are likely unable to do so due to tighter standards. We find that borrowers with LTVs of between 80% and 100% would likely have about \$31 billion in expected real savings over the remainder of the mortgage if they were able to refinance.

29. For example, Driscoll, Downar, and Pilat (1990) provide simple linear models useful in nuclear reactor design. They argue that such models can verify the output from black-box computer simulations.

APPENDIX A: PARTIAL DEDUCTIBILITY OF POINTS

Let F denote the fixed cost of refinancing and $100 \times f$ is the number of points. The expected arrival rate of a full deductibility event—a move or a subsequent refinancing—is θ. At date t, the probability that such a full deductibility event has not yet occurred is $e^{-\theta t}$. The likelihood that such an event occurs at date t is $\theta e^{-\theta t}$.

Assume the term of the new mortgage is for N years. Each year, borrowers are allowed to deduct amount $\frac{fM}{N}$ from their income, producing a tax reduction of $\tau f M/N$. At the time of a full deductibility event, borrowers immediately deduct all of the remaining undeducted points—that is, they reduce their taxes by $\tau f M(N - T)/N$.

The real value of the deduction declines at the rate of inflation. Hence, the payments are discounted effectively at the real discount rate $r = \rho + \pi$.

The present value of these tax benefits is then

$$\int_0^N e^{-\theta t} e^{-(\rho+\pi)t} \left(\frac{\tau f M}{N}\right) dt + \int_0^N \theta e^{-\theta t} e^{-(\rho+\pi)t} (\tau f M) \left(1 - \frac{t}{N}\right) dt. \tag{A1}$$

Using integration by parts, this simplifies to

$$\frac{\tau f M}{\theta + \rho + \pi} \left[\left(\frac{1 - e^{-(\theta+\rho+\pi)N}}{N}\right)\left(\frac{\rho + \pi}{\theta + \rho + \pi}\right) + \theta\right]. \tag{A2}$$

Hence, total refinancing costs $\kappa(M)$ are given by

$$\kappa(M) = F + f M \left[1 - \frac{\tau}{\theta+\rho+\pi}\left[\left(\frac{1 - e^{-(\theta+\rho+\pi)N}}{N}\right)\left(\frac{\rho + \pi}{\theta + \rho + \pi}\right) + \theta\right]\right], \tag{A3}$$

where $\kappa(M)$ is defined as the present value of the cost of refinancing, net of future tax benefits. To calibrate this formula, set $\theta \simeq \mu + 0.10$, where μ is the hazard rate of moving and 0.10 is the (approximate) hazard rate of future refinancing. The actual hazard rate of refinancing will be time varying.

APPENDIX B: TWO LEMMAS

LEMMA 2. *The boundary conditions for R are given by*

$$\begin{aligned} R(x^*) &= R(0) - C(M) - \frac{x^* M}{\rho + \lambda} \\ R'(x^*) &= -\frac{M}{\rho + \lambda} \\ \lim_{x \to \infty} R(x) &= 0. \end{aligned} \tag{B1}$$

PROOF. We derive these from the boundary conditions on V. The value-matching and smooth-pasting conditions at *refinancing boundary* x^* are

$$V(r_0, r, \pi_0, \pi) = V(r, r, \pi, \pi) + C(M). \tag{B2}$$

$$\frac{M}{\rho + \lambda} = \frac{\partial V(r_0, r, \pi_0, \pi)}{\partial r}.$$

Since $V(r_0, r, \pi_0, \pi) = Z(r_0, r, \pi_0, \pi) - R(x^*)$, substitution into the first equation implies

$$Z(r_0, r, \pi_0, \pi) - R(x^*) = Z(r, r, \pi, \pi) - R(0) + C(M). \tag{B3}$$

Rearranging the expression and simplifying the Z terms yields

$$R(x^*) = R(0) - C(M) + \frac{(i_0 - \pi + \lambda)\, M}{\rho + \lambda} - \frac{(r + \lambda)\, M}{\rho + \lambda}.$$

$$= R(0) - C(M) - \frac{x^* M}{\rho + \lambda}. \tag{B4}$$

The value-matching equation states that the value of the program just before refinancing, $V(r_0, r, \pi_0, \pi)$, equals the sum of the value of the program just after refinancing *and* the cost of refinancing, $V(r, r, \pi, \pi) + C(M)$.

Changes in the interest rate (below the refinancing point) do not change the option value terms since the consumer is going to instantaneously refinance anyway. So a rise in the interest rate only increases the NPV of future interest payments. This differential property must be continuous at the boundary ("smooth pasting"), so $R'(x^*) = -M/(\rho + \lambda)$.

The asymptotic boundary condition (for R) is

$$\lim_{x^* \to \infty} R(x^*) = 0. \tag{B5}$$

As the difference between the current nominal interest rate and the original rate on the mortgage grows beyond bound, the value of refinancing goes to zero.

LEMMA 3.[30] *If W is the principal branch of the Lambert W and $k + y \leq 1$-function, then*

$$e^y - y = k \tag{B6}$$

iff

$$y = -k - W(-e^{-k}). \tag{B7}$$

30. We are grateful to Fan Zhang for pointing this result out to us.

PROOF. Lambert's W is the inverse function of $f(x) = xe^x$, so

$$z = W(z)e^{W(z)}. \tag{B8}$$

Let $z = -e^{-k}$, then

$$e^{-k} = -W(-e^{-k})e^{W(-e^{-k})}. \tag{B9}$$

Divide by $e^{W(-e^{-k})}$ and add k to yield

$$e^{[-k-W(-e^{-k})]} - \left[-k - W(-e^{-k})\right] = k. \tag{B10}$$

Hence, $y = -k - W(-e^{-k})$ is the solution to $e^y - y = k$.

APPENDIX C: ANALYTIC DERIVATIVES

This section provides analytic derivatives for the optimal refinancing threshold x^* with respect to the parameters τ, ρ, λ, σ, and κ/M.

$$\frac{\partial x^*}{\partial \tau} = -\frac{(\rho+\lambda)\kappa/M}{(1-\tau)^2}\frac{1}{1+W(-\exp(-\phi))} \tag{C1}$$

$$\begin{aligned}\frac{\partial x^*}{\partial \rho} = &-\frac{3\kappa/M}{2(1-\tau)}\frac{1}{1+W(-\exp(-\phi))}\\ &+\frac{\sigma}{2\sqrt{2}(\rho+\lambda)^{3/2}}[\phi + W(-\exp(-\phi)]\end{aligned} \tag{C2}$$

$$\begin{aligned}\frac{\partial x^*}{\partial \lambda} = &-\frac{3\kappa/M}{2(1-\tau)}\frac{1}{1+W(-\exp(-\phi))}\\ &+\frac{\sigma}{2\sqrt{2}(\rho+\lambda)^{3/2}}[\phi + W(-\exp(-\phi)]\end{aligned} \tag{C3}$$

$$\begin{aligned}\frac{\partial x^*}{\partial \sigma} = &+\frac{(\rho+\lambda)\kappa/M}{\sigma(1-\tau)}\frac{1}{1+W(-\exp(-\phi))}\\ &-\frac{1}{\sqrt{2(\rho+\lambda)}}[\phi + W(-\exp(-\phi)]\end{aligned} \tag{C4}$$

$$\frac{\partial x^*}{\partial(\kappa/M)} = -\frac{(\rho+\lambda)}{(1-\tau)}\frac{1}{1+W(-\exp(-\phi))}. \tag{C5}$$

APPENDIX D: FORMULA FOR λ

Assume that a mortgage is characterized by a constant nominal payment, p, with a nominal interest rate i_0. The remaining nominal principal, N, is given by

$$\dot{N} = -p + i_0 N. \tag{D1}$$

The boundary conditions are $N(0) = N_0$ and $N(T) = 0$. The solution to this differential equation is

$$N(t) = \frac{p}{i_0} + \left(N_0 - \frac{p}{i_0}\right)\exp(i_0 t). \tag{D2}$$

Exploiting the boundary condition at T, we have

$$\begin{aligned} 0 &= N(T) \\ &= \frac{p}{i_0} + \left(N_0 - \frac{p}{i_0}\right)\exp(i_0 T). \end{aligned} \tag{D3}$$

This implies that the nominal payment stream is given by

$$p = \frac{i_0 N_0}{1 - \exp(-i_0 T)}. \tag{D4}$$

We can also show that

$$\begin{aligned} \frac{N(t)}{N_0} &= \frac{p}{N_0 i_0} + \left(1 - \frac{p}{N_0 i_0}\right)\exp(i_0 t) \\ &= \frac{1 - \exp(i_0\,[t - T])}{1 - \exp(-i_0 T)}. \end{aligned} \tag{D5}$$

Hence,

$$\begin{aligned} \frac{p}{N(t)} &= \frac{i_0}{1 - \exp(-i_0 T)} \cdot \frac{N_0}{N(t)} \\ &= \frac{i_0}{1 - \exp(i_0\,[t - T])}. \end{aligned} \tag{D6}$$

So the rate of real repayment at date t is

$$\begin{aligned} \lambda &= \mu + \frac{p}{N} - i_0 + \pi \\ &= \mu + \frac{i_0}{1 - \exp(-i_0 \Gamma)} - i_0 + \pi \\ &= \mu + \frac{i_0}{\exp(i_0 \Gamma) - 1} + \pi, \end{aligned} \tag{D7}$$

where μ is the hazard of moving, Γ is the number of remaining years on the mortgage, and π is the current inflation rate.

APPENDIX E: STANDARD DEVIATION CALCULATIONS

E.1 Chen and Ling's (1989) Assumptions

Chen and Ling (1989) assume that the short rate x_t follows the binomial process

$$\frac{x_{t+1}}{x_t} = \epsilon_{t+1}, \tag{E1}$$

where

$$\epsilon_{t+1} = \begin{cases} U & w/prob\,\pi \\ D & w/prob\,1-\pi. \end{cases} \tag{E2}$$

With constant π, over time the logarithm of x_N/x_0 will follow a binomial distribution with an N-period mean of

$$\mu = N[\pi \ln(U) + (1-\pi)\ln(D)] \tag{E3}$$

and variance

$$\sigma^2 = N[(\ln(U) - \ln(D))^2 \pi(1-\pi)]. \tag{E4}$$

The above expressions for μ and σ can be jointly solved for values of $U > D$ in terms of μ, σ, π, and N:

$$U = \exp\left(\frac{\mu}{N} + \frac{\sigma(1-\pi)}{\sqrt{N\pi(1-\pi)}}\right) \text{ and } D = \exp\left(\frac{\mu}{N} - \frac{\sigma\pi}{\sqrt{N\pi(1-\pi)}}\right). \tag{E5}$$

As $N \to \infty$, this log binomial distribution approaches a log normal distribution.

Chen and Ling use this log-normal approximation to calibrate values of μ and σ from monthly data on 3-month Treasury bills. They choose values for μ of –0.02, 0, and 0.02 and for σ of 5%, 15%, and 25%.

They use the local expectations hypothesis to compute the values of other securities as needed.

E.2 Current Paper's Assumptions

We assume that the 30-year mortgage rate follows a driftless Brownian motion. We calibrate the variance with monthly data on the first difference of Freddie Mac's 30-year mortgage rate series from 1971 to 2004, finding an annualized value of 0.000119.

E.3 Implications of Chen and Ling's (1989) Assumptions for the Current Paper

We can use the log version of the expectations hypothesis to approximate the yield of longer-term securities from Chen and Ling's (1989) short-rate assumptions.

For any security of term s, the log yield of that security approximately satisfies:

$$\ln x_t^s = \frac{1}{s} E_t \left(\ln x_t^1 + \ln x_{t+1}^1 + \ln x_{t+2}^1 + \cdots + \ln x_{t+s-1}^1\right). \tag{E6}$$

In each case, the superscript denotes the term of the security. Hence, the log yield on an s-period security is the average of the expected log yields on the future sequence of s one-period securities.

Under Chen and Ling's assumptions,

$$\ln x_{t+1}^1 = \ln x_t^1 + \ln \epsilon_{t+1}, \tag{E7}$$

where

$$\ln \epsilon_{t+1} = \begin{cases} \dfrac{\mu}{N} + \dfrac{\sigma(1-\pi)}{\sqrt{N\pi(1-\pi)}} & w/prob\, \pi \\ \dfrac{\mu}{N} - \dfrac{\sigma\pi}{\sqrt{N\pi(1-\pi)}} & w/prob\, 1-\pi. \end{cases} \tag{E8}$$

Hence

$$\begin{aligned} \ln x_{t+i}^1 &= \ln x_t^1 + \ln \epsilon_{t+i} + \ln \epsilon_{t+i-1} + \cdots + \ln \epsilon_{t+1} \\ &= \ln x_t^1 + \sum_{j=1}^{i} \ln \epsilon_{t+j}, \end{aligned} \tag{E9}$$

and

$$E_t \ln x_{t+i}^1 = E_t \ln x_t^1 + \sum_{j=1}^{i} \ln \epsilon_{t+j} = \ln x_t^1 + \sum_{j=1}^{i} E_t \ln \epsilon_{t+j}. \tag{E10}$$

Using the assumptions above about how $\ln \epsilon_{t+1}$ evolves,

$$E_t \ln \epsilon_{t+j} = \pi\left(\frac{\mu}{N} + \frac{\sigma(1-\pi)}{\sqrt{N\pi(1-\pi)}}\right) + (1-\pi)\left(\frac{\mu}{N} - \frac{\sigma\pi}{\sqrt{N\pi(1-\pi)}}\right) = \frac{\mu}{N}. \tag{E11}$$

Thus,

$$E_t \ln x_{t+i}^1 = \ln x_t^1 + \sum_{j=1}^{i} \frac{\mu}{N} = \ln x_t^1 + i\,\frac{\mu}{N}. \tag{E12}$$

This implies

$$\begin{aligned} \ln x_t^s &= \frac{1}{s}\left(\ln x_t^1 + \left(\ln x_t^1 + \frac{\mu}{N}\right) + \left(\ln x_t^1 + 2\frac{\mu}{N}\right) \cdots + \left(\ln x_t^1 + (s-1)\frac{\mu}{N}\right)\right) \\ &= \ln x_t^1 + \frac{1}{s}\frac{\mu}{N}\sum_{k=1}^{s-1} k \\ &= \ln x_t^1 + \frac{1}{s}\frac{\mu}{N}\frac{s(s-1)}{2} \\ &= \ln x_t^1 + \frac{\mu}{N}\frac{(s-1)}{2}. \end{aligned} \tag{E13}$$

Thus the first difference of the level of the yield is

$$x_{t+1}^s - x_t^s = e^{\frac{\mu}{N}\frac{(s-1)}{2}}\left(x_{t+1}^1 - x_t^1\right) \equiv K\left(x_{t+1}^1 - x_t^1\right). \tag{E14}$$

Define $\Delta x_{t+1}^s \equiv x_{t+1}^s - x_t^s$. Then

$$\begin{aligned} E_t \Delta x_{t+1}^s &= E_t K\left(x_{t+1}^1 - x_t^1\right) \\ &= K E_t\left(x_{t+1}^1 - x_t^1\right) \\ &= K E_t((\epsilon_{t+1} - 1)x_t^1) \\ &= K x_t^1\left(\pi \exp\left(\frac{\mu}{N} + \frac{\sigma(1-\pi)}{\sqrt{N\pi(1-\pi)}}\right)\right. \\ &\qquad \left. + (1-\pi)\exp\left(\frac{\mu}{N} - \frac{\sigma\pi}{\sqrt{N\pi(1-\pi)}}\right) - 1\right) \end{aligned} \tag{E15}$$

and

$$\begin{aligned} \mathrm{Var}_t \Delta x_{t+1}^s &= \mathrm{Var}_t K\left(x_{t+1}^1 - x_t^1\right) \\ &= K^2 \mathrm{Var}_t((\epsilon_{t+1} - 1)x_t^1) \\ &= K^2\left(x_t^1\right)^2 \mathrm{Var}_t(\epsilon_{t+1}) \\ &= K^2\left(x_t^1\right)^2 \pi(1-\pi)\left(\exp\left(\frac{\mu}{N} + \frac{\sigma(1-\pi)}{\sqrt{N\pi(1-\pi)}}\right)\right. \\ &\qquad \left. -\ \exp\left(\frac{\mu}{N} - \frac{\sigma\pi}{\sqrt{N\pi(1-\pi)}}\right)\right)^2. \end{aligned} \tag{E16}$$

Assume $\pi = \frac{1}{2}$. $N = 12$. Although Chen and Ling assume several different values of μ, for our own specification we assume lack of drift. Setting $\mu = 0$ and $\pi = \frac{1}{2}$ implies $K = 1$ and simplifies the expressions for the conditional mean and variance considerably:

$$\begin{aligned} E_t \Delta x_{t+1}^s &= x_t^1\left(\frac{1}{2}\left(\exp\frac{\sigma}{\sqrt{12}} + \exp-\frac{\sigma}{\sqrt{12}}\right) - 1\right) \\ \mathrm{Var}_t \Delta x_{t+1}^s &= \left(x_t^1\right)^2\left(\frac{1}{4}\left(\exp\frac{2\sigma}{\sqrt{12}} + \exp-\frac{2\sigma}{\sqrt{12}}\right) - \frac{1}{2}\right). \end{aligned} \tag{E17}$$

Note that given the absence of drift, these expressions do not depend on the term s of the security.

Chen and Ling start their short rate at $x_t^1 = 0.08$. For the values of $\sigma = \{0.05, 0.15, 0.25\}$ assumed by Chen and Ling, the corresponding mean, variance, and standard deviation for the 30-year mortgage rate, annualized, are then:

σ	Mean	Variance	Standard deviation
0.05	0.0001	0.000016	0.0040
0.15	0.0009	0.000144	0.0120
0.25	0.0025	0.000401	0.0200

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